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An ergodic theorem for quantum counting processes

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Abstract
For a quantum-mechanical counting process we show ergodicity, under the condition that the underlying open quantum system approaches equilibrium in the time mean. This implies equality of time average and ensemble average for correlation functions of the detection current to all orders and with probability 1.

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1. Introduction

Modern research on quantum-mechanical counting processes, be it numerical simulations [Car] or experimental investigations [MYK], usually starts from the tacit assumption that for the study of statistical properties of the counting records it does not make a difference whether a large number of experiments is performed or a single very long one. This assumption amounts to ergodicity of these records. In several recent discussions, e.g. [BESW, NaS, PlK, Cre, DCM], investigators have addressed the question of its validity, which has never been proved in the quantum-mechanical context. A partial result was obtained by Cresser [Cre], who proved ergodicity in the L^2-sense and to first order in the detection current. In this paper we establish ergodicity in the full sense (theorem 3), under a condition of convergence to equilibrium of the semigroup of completely positive maps describing the dynamics of the source system. Our result holds in particular to all orders in the detection current and exploits Birkhoff's individual ergodic theorem to obtain convergence with probability 1 (theorem 4). Theorem 5 formulates ergodicity in terms of multi-time coincidences.

For the description of detection records we employ the rigorous formulation of Davies and Srinivas [Dav, SrD], which has set the tone for later investigations [Car, WiM, GaZ].

2. Counting processes according to Davies and Srinivas

We consider an open quantum system under continuous observation by use of a finite number k of detectors. The state of the system is described by a density matrix ρ on a Hilbert space, obeying a master equation $\dot{\rho} = L\rho$, where L is a generator of Lindblad form [Lin]. Normalization is expressed by the relation

$$\operatorname{tr} L(\rho) = 0 \qquad \text{for all} \quad \rho. \tag{2.1}$$

A counting process connected to this quantum evolution is based on an unravelling of the generator

$$L = L_0 + \sum_{i=1}^{k} J_i \tag{2.2}$$

which is interpreted as follows. The reaction of the detectors to the system consists of clicks at random times. The evolution $\rho \mapsto \mathrm{e}^{tL_0}(\rho)$ denotes the change of the state of the system under the condition that during a time interval of length t no clicks are recorded. The operator $\rho \mapsto J_i(\rho)$ on the state space describes the change of state conditioned on the occurrence of a click of detector i. For computational convenience we assume these operators to be bounded. So, if ρ describes the state of the system at time 0, and if, during the time interval $[0, t]$, clicks are recorded at times $t_1, t_2, \ldots, t_n$ of detectors $i_1, i_2, \ldots, i_n$ respectively, and none more, then, up to normalization, the state at time t is given by

$$\mathrm{e}^{(t-t_n)L_0} J_{i_n} \mathrm{e}^{(t_n-t_{n-1})L_0} \cdots \mathrm{e}^{(t_2-t_1)L_0} J_{i_1} \mathrm{e}^{t_1 L_0}(\rho). \tag{2.3}$$

The probability density $f^t((t_1, i_1), \ldots, (t_n, i_n))$ for these clicks to occur is equal to the trace of (2.3).

We imagine the experiment to continue indefinitely. The observation process will then produce an infinite detection record $((t_1, i_1), (t_2, i_2), (t_3, i_3), \ldots)$, where we assume that $0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant \ldots$, and $\lim_{n\to\infty} t_n = \infty$ (i.e., the clicks do not accumulate).

Let Ω denote the space of all such detection records. By an *event* we mean some property of the record, which we identify with the set $E \subset \Omega$ of all records with this property. The events decidable at or before time $t \geqslant 0$ form a σ-algebra Σ_t [Dav]. Together these σ-algebras generate the full σ-algebra Σ. Following Davies and Srinivas we may now formulate the effect of observation on the quantum system as follows: if t is a positive time, E an event in Σ_t, and ρ denotes a state, then we define

$$M_t(E)(\rho) := \sum_{n=0}^{\infty} \sum_{i_1=1}^{k} \cdots \sum_{i_n=1}^{k} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} 1_E((t_1, i_1), \ldots, (t_n, i_n))\, \mathrm{e}^{(t-t_n)L_0} J_{i_n} \mathrm{e}^{(t_n-t_{n-1})L_0} \ldots \\ \times \mathrm{e}^{(t_2-t_1)L_0} J_{i_1} \mathrm{e}^{t_1 L_0}(\rho)\, \mathrm{d}t_1\, \mathrm{d}t_2 \cdots \mathrm{d}t_n. \tag{2.4}$$

Here 1_E denotes the indicator function of the event E and $M_t(E)$ is the effect on the quantum system of the occurrence of $E \in \Sigma_t$. Then

$$\mathbb{P}_\rho^t(E) := \operatorname{tr} M_t(E)(\rho) \tag{2.5}$$

is the probability of the occurrence of E given that the system starts in ρ. We extend the notation (2.5) also to density matrices ρ which are not normalized. The counting process as a whole is described by the family $(M_t)_{t\geqslant 0}$. The effect of the counting on the quantum system, when the outcome is ignored, is the time evolution

$$T_t(\rho) := M_t(\Omega)(\rho).$$

It follows from the Dyson series (2.4) with $E = \Omega$ that T_t is indeed the original time evolution e^{tL}, in particular, by (2.1), T_t preserves the trace.

3. Ergodic theory

The time shift by t seconds is described by the map σ_t on Ω, which is given on a particular record $\omega = ((t_1, i_1), (t_2, i_2), (t_3, i_3), \ldots) \in \Omega$ with $t_k \leqslant t < t_{k+1}$ by $\sigma_t(\omega) := ((t_{k+1} - t, i_{k+1}), (t_{k+2} - t, i_{k+2}), \ldots)$. The time shift of an event E towards the future is given by $\sigma_t^{-1}(E)$.

The crucial property of the counting process $(M_t)_{t\geqslant 0}$ is the following. For all $s, t \geqslant 0$ and all events $E \in \Sigma_s$, $F \in \Sigma_t$ we have

$$M_{s+t}\big(F \cap \sigma_t^{-1}(E)\big) = M_s(E) \circ M_t(F). \tag{3.1}$$

This Markov property was proved in [Dav]. Putting $E = F = \Omega$ we recover the semigroup property $T_{s+t} = T_s \circ T_t$ of the time evolution.

When $F \in \Sigma_t$ and $s \geqslant 0$ then $\mathbb{P}_\rho^{t+s}(F)$ does not depend on s. Indeed, since $\Omega = \sigma_t^{-1}(\Omega)$ and T_s preserves the trace,

$$\begin{aligned}\mathbb{P}_\rho^{t+s}(F) &= \operatorname{tr}(M_{t+s}(F)(\rho)) = \operatorname{tr}\big(M_{t+s}\big(F \cap \sigma_t^{-1}(\Omega)\big)(\rho)\big)\\ &\overset{(3.1)}{=} \operatorname{tr}(M_s(\Omega) \circ M_t(F)(\rho)) = \operatorname{tr}(T_s \circ M_t(F)(\rho))\\ &= \operatorname{tr}(M_t(F)(\rho)) = \mathbb{P}_\rho^t(F).\end{aligned}$$

Therefore, by Kolmogorov's extension theorem, the family $\big(\mathbb{P}_\rho^t\big)_{t\geqslant 0}$ of probability measures on the σ-algebras $(\Sigma_t)_{t\geqslant 0}$ with densities $(f^t)_{t\geqslant 0}$ extends to a single probability measure $\mathbb{P}_\rho$ on the full σ-algebra Σ.

Lemma 1. *For all $t \geqslant 0$, all $E \in \Sigma$, $F \in \Sigma_t$ and all states ρ:*

$$\mathbb{P}_\rho\big(F \cap \sigma_t^{-1}(E)\big) = \mathbb{P}_{M_t(F)(\rho)}(E). \tag{3.2}$$

In particular,

$$\mathbb{P}_\rho\big(\sigma_t^{-1}(E)\big) = \mathbb{P}_{T_t\rho}(E). \tag{3.3}$$

Therefore, if ρ is invariant under T_t, then $\mathbb{P}_\rho$ is a stationary probability measure on Ω.

Proof. First suppose that $E \in \Sigma_s$. Equality (3.2) is obtained from the Markov property (3.1) by acting on ρ and taking the trace on both sides. Equation (3.3) follows by putting $F = \Omega$. The statements extend to all $E \in \Sigma$ by Kolmogorov's extension theorem since s was arbitrary. □

Definition.

- *The evolution $(T_t)_{t\geqslant 0}$ of a quantum system is said to converge in the mean to an equilibrium state ρ if for all normalized density matrices ϑ and all observables x:*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \operatorname{tr}((T_t\vartheta)x)\,\mathrm{d}t = \operatorname{tr}(\rho x).$$

- *The counting process $(M_t)_{t\geqslant 0}$ will be called ergodic if the following holds. Given any time-invariant event E, i.e. $\sigma_t^{-1}(E) = E$ for all $t \geqslant 0$, then either $\mathbb{P}_\vartheta(E) = 0$ for all density matrices ϑ or $\mathbb{P}_\vartheta(E) = 1$ for all ϑ.*

The condition on $(T_t)_{t\geqslant 0}$ is satisfied in many cases of practical importance. In particular if $(T_t)_{t\geqslant 0}$ has only one normal equilibrium state and either multiples of the unit operator are the only invariant observables or $\mathcal{H}$ is finite dimensional, then the above convergence in the mean automatically holds.

Theorem 2. *If the evolution* $T_t = e^{tL}$, $t \geqslant 0$, *converges in the mean, then the counting process* $(M_t)_{t\geqslant 0}$ *is ergodic for any unravelling (2.2).*

Proof. Let E be a time-invariant event and ϑ any state. Then by (3.3), $\mathbb{P}_\vartheta(E) = \mathbb{P}_\vartheta\big(\sigma_t^{-1}(E)\big) = \mathbb{P}_{T_t\vartheta}(E)$. Since $\mathbb{P}_\vartheta$ is linear and continuous in ϑ, we may average both sides over the interval $[0, \tau]$ and take the limit $\tau \to \infty$ to obtain $\mathbb{P}_\vartheta(E) = \mathbb{P}_\rho(E)$. For an unnormalized density matrix χ we find instead that

$$\mathbb{P}_\chi(E) = \mathbb{P}_\rho(E)\mathrm{tr}(\chi). \tag{3.4}$$

If F is any event in Σ_t then

$$\begin{aligned}\mathbb{P}_\vartheta(F\cap E) &= \mathbb{P}_\vartheta\big(F\cap\sigma_t^{-1}(E)\big) \overset{(3.2)}{=} \mathbb{P}_{M_t(F)(\vartheta)}(E)\\ &\overset{(3.4)}{=} \mathbb{P}_\rho(E)\mathrm{tr}(M_t(F)(\vartheta)) = \mathbb{P}_\rho(E)\mathbb{P}_\vartheta(F) \overset{(3.4)}{=} \mathbb{P}_\vartheta(E)\mathbb{P}_\vartheta(F).\end{aligned}$$

The resulting equation extends to all $F \in \Sigma$, in particular it holds for $F = E$:

$$\mathbb{P}_\vartheta(E) = \mathbb{P}_\vartheta(E)^2.$$

It follows that $\mathbb{P}_\vartheta(E)$ is equal to 0 or 1. $\square$

Let us denote the expectation $\int_\Omega f(\omega)\,\mathrm{d}\mathbb{P}_\rho(\omega)$ of an integrable function f on Ω by $\mathbb{E}_\rho(f)$.

Theorem 3. *If the evolution* $(T_t)_{t\geqslant 0}$ *converges in the mean to* ρ, *then for all integrable functions* h *on* Ω *and all initial states* ϑ *we have, almost surely with respect to* $\mathbb{P}_\vartheta$,

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau h(\sigma_t(\omega))\,\mathrm{d}t = \mathbb{E}_\rho(h). \tag{3.5}$$

Proof. By lemma 1 and theorem 2, $\mathbb{P}_\rho$ is stationary and ergodic. Hence, by Birkhoff's individual ergodic theorem, the limit on the left exists almost surely with respect to $\mathbb{P}_\rho$, and is equal to the constant $\mathbb{E}_\rho(h)$. Since the set F of points $\omega \in \Omega$ for which (3.5) holds, is time invariant, we have $\mathbb{P}_\vartheta(F) = \mathbb{P}_\rho(F) = 1$ for all states ϑ by (3.4). $\square$

4. Applications

The main result of the present ergodic theory for quantum counting processes, theorem 3, can be made considerably more concrete by applying it to detection currents and multi-time coincidences. These applications are standard consequences of the ergodicity property.

For simplicity we consider only one detector, which responds to a point event at time s by producing a current $\gamma(t-s)$ at time t. (This will be zero for $t < s$.) The total detection current is given by

$$I_t(\omega) := \sum_{s\in\omega}\gamma(t-s).$$

Let $\widetilde{\mathbb{P}}_\rho$ be the unique stationary extension of $\mathbb{P}_\rho$ to negative times on the configuration space $\widetilde{\Omega}$ of the full real line. We shall denote expectation with respect to this measure by $\widetilde{\mathbb{E}}_\rho$.

Theorem 4. *Let the quantum evolution* $(T_t)_{t\geqslant 0}$ *converge in the mean to a state* ρ *and let the detector response function* $\gamma : \mathbb{R} \to [0, \infty)$ *be bounded and integrable. Then for all* $0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n$ *and all initial states* ϑ *we have, almost surely with respect to* $\mathbb{P}_\vartheta$,

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau I_{t_1+t}(\omega)\cdots I_{t_n+t}(\omega)\,\mathrm{d}t = \widetilde{\mathbb{E}}_\rho\big(I_{t_1}\cdots I_{t_n}\big).$$

For $n = 2$ this theorem implies a quantum-mechanical version of the Wiener–Khinchin theorem which was proved by Cresser in the mean square sense [Cre]. In the proof we shall make use of the non-exclusive probability density of the process [Ram, Bar, Str, vKa, GaZ], which is stationary since $T_t\rho = \rho$ and $\mathrm{tr} \circ T_t = \mathrm{tr}$,

$$g_n(t_1, t_2, \ldots, t_n) := \mathrm{tr}\big(J T_{t_n - t_{n-1}} J \cdots J T_{t_2 - t_1} J(\rho)\big).$$

The advantage of using the functions g_n instead of f_n will be clear from theorem 5: they have a straightforward physical interpretation as n-time correlation functions, which can be measured in disregard of events taking place at other instants of time. The functions g_n are related to the probability densities f_n^t from (2.3) of the counting process (where $t \geqslant t_n$), by

$$g_n(t_1, t_2, \ldots, t_n) = f_n^t(t_1, t_2, \ldots, t_n) + \sum_{m=1}^{\infty} \int_0^t \int_0^{s_m} \cdots \int_0^{s_2} f_{m+n}^t(\{t_1, \ldots, t_m\} \cup \{s_1, \ldots, s_n\})$$
$$\times \, \mathrm{d}s_1 \cdots \mathrm{d}s_m = \int_{\Omega_t} f^t(\{t_1, t_2, \ldots, t_n\} \cup \omega) \, \mathrm{d}\omega \tag{4.1}$$

where Ω_t is the set of finite subsets of $[0, t]$, which can be identified with the time-ordered points in $\{\emptyset\} \cup \bigcup_{m=1}^{\infty} [0, t]^m$. By $\mathrm{d}\omega$ we mean $\mathrm{d}s_1\, \mathrm{d}s_2 \cdots \mathrm{d}s_m$ if $\omega = \{s_1, s_2, \ldots, s_m\}$ with $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_m$.

Proof of theorem 4. First we note that theorem 3 also holds if Ω, $\mathbb{P}_\rho$ and $\mathbb{E}_\rho$ are replaced by $\widetilde{\Omega}$, $\widetilde{\mathbb{P}}_\rho$ and $\widetilde{\mathbb{E}}_\rho$ respectively, as introduced above, and σ_t by the left shift of $\omega \subset \mathbb{R}$. Then we have $I_{s+t}(\omega) = I_s(\sigma_t(\omega))$. Now fix $n \in \mathbb{N}$ and $0 \leqslant t_1 \leqslant \cdots \leqslant t_n$. Let $h : \widetilde{\Omega} \to \mathbb{R}$ be given by

$$h(\omega) := I_{t_1}(\omega) I_{t_2}(\omega) \cdots I_{t_n}(\omega).$$

It follows that $h \circ \sigma_t = I_{t_1+t} I_{t_2+t} \cdots I_{t_n+t}$, and the statement to be proved follows from theorem 3, provided that h is integrable. In the appendix we shall show that this is indeed the case. □

As our second application we shall show that the non-exclusive probability densities g_n have a straightforward pathwise interpretation: they are equal to the frequency of multi-time coincidences on almost every detection record. For this, let $N_{[a,b]}(\omega) := \#(\omega \cap [a, b])$ denote the number of clicks detected during the time interval $[a, b]$.

Theorem 5. *Let $(T_t)_{t \geqslant 0}$ converge in the mean to the equilibrium state ρ. Then for all $n \in \mathbb{N}$, all $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, all ε between 0 and $\min_{1 \leqslant j < n}(t_{j+1} - t_j)$ and all initial states ϑ we have, almost surely with respect to $\mathbb{P}_\vartheta$,*

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \left(\prod_{j=1}^n N_{[t_j+t, t_j+t+\varepsilon]}(\omega) \right) \mathrm{d}t = \int_{t_n}^{t_n+\varepsilon} \cdots \int_{t_1}^{t_1+\varepsilon} g(s_1, \ldots, s_n) \, \mathrm{d}s_1 \cdots \mathrm{d}s_n. \tag{4.2}$$

Proof. Fix $n \in \mathbb{N}$ and a sequence $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ of times. Let $K : \Omega \to \{0, 1\}$ be the function that maps $\omega \in \Omega$ to 1 if ω contains exactly n points, one in each of the intervals $[t_1, t_1 + \varepsilon], \ldots, [t_n, t_n + \varepsilon]$, and to 0 otherwise. Then we obtain for $t \geqslant t_n + \varepsilon$, using set notation and the integral-sum lemma from [LiM],

$$\int_{t_n}^{t_n+\varepsilon} \cdots \int_{t_1}^{t_1+\varepsilon} g(s_1, \ldots, s_n) \, \mathrm{d}s_1 \cdots \mathrm{d}s_n = \int_{\Omega_t} K(\alpha) g(\alpha) \, \mathrm{d}\alpha$$
$$\stackrel{(4.1)}{=} \int_{\Omega_t} \int_{\Omega_t} K(\alpha) f^t(\alpha \cup \beta) \, \mathrm{d}\alpha \, \mathrm{d}\beta \stackrel{[\mathrm{LiM}]}{=} \int_{\Omega_t} \left(\sum_{\alpha \subset \omega} K(\alpha) \right) f^t(\omega) \, \mathrm{d}\omega. \tag{4.3}$$

A short calculation shows that

$$\sum_{\alpha\subset\omega} K(\alpha) = \prod_{j=1}^{n} N_{[t_j,t_j+\varepsilon]}(\omega). \tag{4.4}$$

Since $0 \leqslant g_n(s_1, s_2, \ldots, s_n) \leqslant \|J\|^n$, the integral (4.3) is convergent, hence the product on the rhs of (4.4) is integrable as a function of ω. Application of theorem 3 to this product now yields the statement. □

5. Discrete time

There is an obvious analogue of our main result (theorem 3) in discrete time [MaK]. A Kraus measurement [Kra] is given by a decomposition of a completely positive operator T on state space as

$$T\rho = \sum_{i=1}^{k} a_i \rho a_i^*$$

where $\rho \mapsto a_i\rho a_i^*$ describes the state change of the density matrix ρ when the measurement gives the outcome i. Thus for initial state ϑ the probability of finding the sequence of outcomes $i_1, i_2, \ldots, i_m$ by repeated Kraus measurement is given by

$$\mathrm{tr}\big(a_{i_m}\cdots a_{i_1}\vartheta a_{i_1}^*\cdots a_{i_m}^*\big).$$

As in continuous time, this yields a probability measure $\mathbb{P}_\vartheta$ on the space of detection records $\Omega := \{1, 2, \cdots, k\}^{\mathbb{N}}$. Again, if $(T^n)_{n\in\mathbb{N}}$ converges in the mean to some state ρ, then the only time-invariant events in Ω have measure 0 or 1 for all $\mathbb{P}_\vartheta$. In particular, $\mathbb{P}_\rho$ is ergodic.

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Appendix

We shall show that, in the situation of theorem 4, $h := I_{t_1}\cdots I_{t_n}$ is an integrable function on $\widetilde{\Omega}$ provided that the jump operator J is bounded and the detector response function $\gamma : \mathbb{R} \to [0, \infty)$ is bounded and integrable.

Let $M := \max(1, \|\gamma\|_\infty)$. Fix $n \in \mathbb{N}$ and a sequence $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$ of times. Let

$$\varphi(t) := \sum_{j=1}^{n} \gamma(t_j - t).$$

Then φ is also integrable, with $\|\varphi\|_1 = n\|\gamma\|_1$. For $k \in \mathbb{N}$, let $\mathcal{J}_{n,k}$ denote the set of all surjections $\{1, \ldots, n\} \to \{1, \ldots, k\}$. Then we may write for any $\omega \in \widetilde{\Omega}$,

$$\begin{aligned} I_{t_1}(\omega)I_{t_2}(\omega)\cdots I_{t_n}(\omega) &= \sum_{s_1\in\omega}\cdots\sum_{s_n\in\omega}\gamma(t_1-s_1)\cdots\gamma(t_n-s_n) \\ &= \sum_{k=1}^{n}\sum_{j\in\mathcal{J}_{n,k}}\sum_{\substack{\{a_1,\ldots,a_k\}\subset\omega\\ a_1<\cdots<a_k}}\gamma(t_1-a_{j(1)})\cdots\gamma(t_n-a_{j(n)}) \end{aligned}$$

$$\leqslant \sum_{k=1}^{n} \#\left(\mathcal{J}_{n,k}\right) \sum_{\substack{\alpha \subset \omega \\ \#\alpha = k}} \|\gamma\|_{\infty}^{n-k} \left(\prod_{s \in \alpha} \varphi(s) \right) \leqslant n \cdot n^n M^n \sum_{\alpha \subset \omega} \left(\prod_{s \in \alpha} \varphi(s) \right). \tag{A1}$$

Using set notation and the integral-sum lemma [LiM] again we conclude that, for all $t \geqslant 0$ and $u \geqslant t_n + t$,

$$\mathbb{E}_\rho\big(\big(I_{t_1} I_{t_2} \cdots I_{t_n}\big) \circ \sigma_t\big) \big/ M^n n^{n+1} \overset{(\mathrm{A1})}{\leqslant} \int_{\Omega_u} \sum_{\alpha \subset \omega} \left(\prod_{s \in \alpha} \varphi(s-t) \right) f^u(\omega)\, \mathrm{d}\omega$$

$$\overset{[\mathrm{LiM}]}{=} \int_{\Omega_u} \int_{\Omega_u} \left(\prod_{s \in \alpha} \varphi(s-t) \right) f^u(\alpha \cup \beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta \overset{(4.1)}{=} \int_{\Omega_u} \left(\prod_{s \in \alpha} \varphi(s-t) \right) g(\alpha)\, \mathrm{d}\alpha$$

$$\leqslant \sum_{m=0}^{\infty} \frac{\|J\|^m}{m!} \int_{[0,u]^m} \varphi(s_1 - t) \cdots \varphi(s_m - t)\, \mathrm{d}s_1 \cdots \mathrm{d}s_m$$

$$\leqslant \exp\left(\|J\| \int_0^u \varphi(s-t)\, \mathrm{d}s \right) \leqslant \mathrm{e}^{n\|J\| \cdot \|\gamma\|_1}.$$

Therefore, since the rhs does not depend on t,

$$\widetilde{\mathbb{E}}_\rho\big(I_{t_1} \cdots I_{t_n}\big) = \lim_{t \to \infty} \mathbb{E}_\rho\big(\big(I_{t_1} \cdots I_{t_n}\big) \circ \sigma_t\big) < \infty.$$

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